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The global instability of uniform flows in non-one-dimensional regions☆

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Abstract

The conditions for the instability of flows or states, which are independent of time and coordinates, in extended non-one-dimensional regions are considered in a linear approximation. An extension of the idea of global instability, previously introduced for the one-dimensional case, is given. A method is proposed for weakly unstable flows, which enables one to investigate under what conditions perturbations, which grow without limit with time, and which do not depend on the specific form of boundary conditions (provided they are not degenerate), exist. The case of a two-dimensional rectangular region is considered in detail.

When considering the instability of one-dimensional steady states and flows in a section of large but finite length L, it was found[1] that instability can occur in two forms: (1) in the form of increasing perturbations, the existence of which, and also their natural frequencies, are determined by the boundary conditions at one end of the section ("unilateral", or boundary instability); the natural frequencies of these perturbations approach finite limits as the length of the section, in which the perturbations develop, approaches infinity; (2) in the form of the existence of growing perturbations, which, far from the boundaries, take the form of two waves, travelling in opposite directions and reflected from the ends of the section ("global" instability). The equation giving the natural frequencies corresponding to global instability, takes the following form as the length of the section L approaches infinity

$$\operatorname{Im}k_r(\omega) = \operatorname{Im}k_l(\omega) \tag{0.1}$$

Here $k_r(\omega)$ and $k_l(\omega)$ are branches of the analytical function, which, from the dispersion equation, express a wave-number system as a function of the frequency ω. Here the branches $k_r(\omega)$ and $k_l(\omega)$ correspond to perturbations which, among all the types of perturbations which propagate to the right and, correspondingly, to the left, undergo the greatest spatial increase or the least attenuation. If the length L is large, but finite, the natural frequencies are situated in the complex plane ω at distances from one another and from the line, defined by Eq. (0.1), which decrease as $1/L$ as L increases. Hence, the satisfaction of Eq. (0.1) for $\operatorname{Im}\omega > 0$ is a sufficient condition for instability for large L. When deriving Eq. (0.1) it was assumed that the coefficients of the mutual conversion of perturbations, corresponding to $k_r(\omega)$ and $k_l(\omega)$ are finite, when the perturbations are reflected from the ends of the section, and also, as already stated, as $L \to \infty$. Below we will assume that $L \gg \lambda$, where λ is the characteristic wavelength of the perturbations corresponding to k_r and k_l.

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Hence, for large L the possibility that an eigenfunction will exist is determined by the spatial change in the amplitudes of the two fundamental waves, comprising the "skeleton" of the eigenfunction, while the eigenfrequency ω is found from the condition that for cylic transit of the section of length L by these waves in the forward and reverse directions, the spatial amplification $L\mathrm{Im}(k_r(\omega) - k_l(\omega))$ remains finite.

Similar results were obtained in Ref. 2 when considering one-dimensional perturbations in the case when the boundary conditions are specified not only at the ends of the section, but also at intermediate points. Moreover, degeneration of the boundary conditions, related to the fact that certain reflection or refraction coefficients vanish, were permitted. Global instability in this case also occurs in the form of the existence of a finite train (sequence) of waves, which depend on time as $e^{-i\omega t}$ with $\mathrm{Im}\omega > 0$, which are mutually converted into one another at points where the boundary conditions are specified. The train may consist of two waves, as in the case considered previously in Ref. 1, and may also contain a larger number of waves. The condition from which $\mathrm{Im}\omega$ is found, is due to the fact that the spatial amplification of the waves in all the sections of the train is equal to a reciprocal of the product of all the reflection and refraction coefficients. This condition, like equality (0.1), can be written as a relation between the imaginary parts of the wave numbers of the waves of the train, which depend on ω.

The purpose of the present paper is to generalize the results obtained previously in Refs. 1,2 and to extend the idea of global instability to non-one-dimensional problems.

The problem consists of detecting, in non-one-dimensional extended regions, perturbations which increase with time, and the growth of which, as in one-dimensional problems,[1,2] is determined by the properties of the dispersion equation and, for a characteristic dimension of the region tending to infinity, is independent of the specific form of the boundary conditions, provided they are non-degenerate. Here, as previously in Ref. 1, the boundary conditions are assumed to be independent of the dimensions of the region (when these dimensions tend to infinity) and they are non-degenerate in the sense that the reflection coefficients of the perturbations from the boundaries of the region are non-zero. As previously,[1,2] it is also assumed that the characteristic linear dimension in which any considerable change in the perturbation (the characteristic wavelength) occurs is much less than the characteristic dimension of the region. Instability due to the existence of a growing perturbation of the type described will be called global, by analogy with the terminology employed in Ref. 1,2.

1. The sufficient condition for the existence of growing perturbations in the case of weak instability

For simplicity we will consider the situation when the instability is weak, i.e. when the dispersion equation of the system

$$F(\omega, k_j) = 0$$

is such that, for a real wave vector $\mathbf{k} = \mathbf{e_j}k_j$ ($\mathbf{e_j}$ are the unit vectors of a Cartesian system of coordinates), we have the relation

$$\mathrm{Im}\omega \ll \mathrm{Re}\omega \tag{1.1}$$

In this case we can confine ourselves to considering perturbations in which

$$\mathrm{Im}k_j \ll \mathrm{Re}k_j, \quad j = 1, 2, 3 \tag{1.2}$$

with the simultaneous satisfaction of inequality (1.1).

We will first consider the behaviour of perturbations which have the form of wave packets. When conditions (1.1) and (1.2) are satisfied the wave packet with wave numbers close to certain real values of k_j, propagate with a group velocity $\mathbf{V} = V_i\mathbf{e_j}$, where

$$V_j = \partial\mathrm{Re}\omega/\partial k_j, \quad \mathrm{Im}k_j = 0 \tag{1.3}$$

The complexity of the ω and k_j values will manifest itself in a change in the amplitude of the wave packets, propagating inside the region and which are reflected from its boundaries from time to time. We will assume that the dimension L of the region is so large that the characteristic value of the logarithm of the amplification of a packet, when the distance it traverses is of the order L, is much greater in modulus that the logarithm of the typical reflection coefficient. In other words, we will assume that amplification and attenuation of the waves occurs mainly when they are propagating, and is not due to reflections. Obviously the fact that the group velocity vanishes when $\mathrm{Im}\omega > 0$ leads to an unlimited growth of the perturbations. This growth, when there are no boundaries, corresponds to absolute instability, which obviously gives rise to instability in a larger bounded region.

With assumptions (1.1) and (1.2), the boundary conditions can be regarded as the conditions for waves with real frequencies ω to be reflected, neglecting the influence of the imaginary parts. As previously,[1] we will assume that all the reflection coefficients of the waves are finite. From one reflection to another the wave packets propagate inside the region in straight lines, the direction of which is determined by the group velocities.

When wave packets are reflected, different versions are possible. We will confine ourselves to the case of specular reflection, when an arriving perturbation, corresponding to a definite frequency and direction of the wave vector, is reflected in the form of a single or several waves of different kinds, corresponding to the same frequency. Hence, we can speak of sequences trains of waves, which are converted into one another on reflection. As previously in Ref. 2, we will mean by a train of waves a uniquely extendable sequence of them. When reflection of an incident wave occurs in the form of several waves we will speak of the existence of several trains, corresponding to the incident wave with different extensions.

A perturbation propagation trajectory in the form of a broken line will correspond to a train (sequence) of waves. The trajectories may be closed (periodic) and open (infinite). For each trajectory one can calculate the rate of growth (or attenuation) of the amplitude of the perturbations with time.

We will denote by $\mathrm{Im}\omega_\alpha$ the time-dependent increment of the amplitude of a wave packet when it crosses a section of the broken line with number α (here and henceforth Greek subscripts will denote the number of the section of the trajectory). If the trajectory is closed, we can introduce an average increment of growth during the time it takes for the wave packet to go round the whole trajectory

$$\mathrm{Im}\omega = \sum T_\alpha \mathrm{Im}\omega_\alpha / \sum T_\alpha, \quad T_\alpha = l_\alpha / V_\alpha \tag{1.4}$$

Here and everywhere henceforth, unless otherwise stated, the summation is carried out over all the numbers α, l_α is the length of a section of the broken line, and V_α is the modulus of the group velocity on the α-th section of the broken line (the direction of the group velocities is identical with the direction of the sections of the broken line).

As has already been stated, when reflection occurs the frequency ω remains unchanged, and when wave packets are reflected it is sufficient to require that the values of $\mathrm{Re}\omega$ are conserved.

The $\mathrm{Im}\omega$ value can also be obtained by a somewhat different method, similar to the one used in the one-dimensional case.[1,2]

We will consider waves corresponding to a certain fixed complex frequency ω, occupying the whole trajectory. We will find ω for closed trajectories (polygons) from the requirement that the wave amplitude should be a unique function of a point on the trajectory. From the condition for the amplitude to be unique, neglecting the difference of the reflection coefficients from unity, we obtain

$$\sum l_\alpha \mathrm{Im} k_\alpha(\omega_0 + \Delta\omega) = 0; \quad \mathrm{Im}\omega_0 = 0, \quad \mathrm{Re}\Delta\omega = 0 \tag{1.5}$$

where $\omega = \omega_0 + \Delta\omega$ is the required frequency of the train of waves and k_α is the projection of the vector $\mathbf{k}$ onto the direction of propagation along the α-th side of the broken line. The quantity $\mathrm{Im}k_\alpha$ represents the spatial increment of the amplitude along the α-th side.

Since $\Delta\omega$ is a small quantity, taking relations (1.3) into account, we can write (1.5) in the form

$$\mathrm{Im}\omega = -\sum l_\alpha \mathrm{Im} k_\alpha(\omega_0) / \sum l_\alpha V_\alpha^{-1} \tag{1.6}$$

Actually, we will consider the identity

$$k_\alpha(\omega_0 + \Delta\omega) = k_\alpha(\omega_0 + \Delta_\alpha\omega + \Delta\omega - \Delta_\alpha\omega)$$

where $\mathrm{Im}\omega_0 = 0$, and we will define the quantity $\Delta_\alpha\omega$ by the condition $\mathrm{Im}k_\alpha(\omega_0 + \Delta_\alpha\omega) = 0$. The latter denotes that $\omega_0 + \Delta_\alpha\omega$ has the same meaning as ω_α in Eq. (1.4). By virtue of relations (1.1) and (1.2) the quantities $\Delta\omega$ and $\Delta_\alpha\omega$ can be assumed to be small. Hence,

$$\mathrm{Im} k_\alpha(\omega_0 + \Delta\omega) = \frac{1}{V_\alpha}\mathrm{Im}(\Delta\omega - \Delta_\alpha\omega); \quad V_\alpha = \frac{\partial\omega}{\partial k_\alpha}$$

Substituting this expression into (1.5) and taking into account the fact that

$$\mathrm{Im}\omega = \mathrm{Im}\Delta\omega, \quad \mathrm{Im}\omega_\alpha = \mathrm{Im}\Delta_\alpha\omega$$

we obtain equality (1.4). Hence, on closed trajectories the increment of the perturbation growth rate can be found both from the requirement for the amplitude along the trajectory to be unique, and by taking the time averages of the increments of the growth of the wave packets along the sections of the trajectory.

If the trajectory is open, we must average the perturbation growth rate over a longer time interval and take the limit as the averaging time tends to infinity.

If a trajectory exists on which the perturbation grows, this means that the state investigated is unstable. In fact, a trajectory with the most rapid growth of perturbations exists. Perturbations related to other trajectories will grow more slowly with time and cannot prevent the growth of the most rapidly growing perturbation. Possible losses, due to spatial spreading of the beams of wave propagating along neighbouring beams, may lead to a reduction in the amplitudes, which have the form of inverse powers of the path length, traversed by the perturbation. For fairly large distances, the exponential growth of the amplitude of a perturbation, due to instability, considerably surpasses the effect of the power attenuation related to the divergence of the beam. Hence, the amplification of the perturbations, which is determined using some train of waves, leads, for sufficiently large L, to the occurrence of growing perturbations and, consequently, to the conclusion that there is instability.

The perturbations discussed above are similar in some respects to acoustic oscillations between two reflecting surfaces.[3] In order for such oscillations to travel without attenuation it is necessary that the reflecting surfaces should possess concavity, which prevents the loss of energy from the region where the oscillations occur. In the case considered, when the waves may be amplified exponentially while they are propagating, the curvatures of the reflecting surfaces may have any signs, which does not prevent a growth of the perturbations as they propagate along the trajectories when the lengths of the trajectories are considerable.

We will now consider the problem with the initial data, concentrated in a certain region with a characteristic dimension $l \ll L$, and situated fairly far from the boundaries. If we neglect the boundary effect, this perturbation will decompose, after a certain time, into waves propagating with their own group velocities.[4] This enables us to assume that the condition considered above for the amplitude of any of the wave packets to grow, taking into account their reflections from the boundaries, represents a general global instability, i.e. instabilities which occur for large L and are independent of the particular specification of the boundary conditions.

Hence, the problem of detecting global instability reduces to finding the trajectories along which growth of the perturbations occurs, i.e. trajectories with $\mathrm{Im}\omega > 0$. This approach to finding the condition for global instability is illustrated below using the example of the development of perturbations with a number of additional specifying assumption.

Note that an approach similar to that described above has been used previously to prove the instability of a number of non-uniform steady flows of a viscous incompressible fluid.[5,6] The trajectories along which the perturbations transfer and grow are represented in this case by closed streamlines. There is also a relation between the method described above and the Hamilton-Jacobi complex equations method,[7,8] according to which the evolution of perturbations on a slightly non-uniform background is found by integrating the above-mentioned equations.

2. The conditions for the growth of perturbations in a rectangular region

We will consider the problem of the growth of perturbations in a large rectangular (two-dimensional) region when conditions (1.1) and (1.2) are satisfied. In addition we will assume that the waves which participate in the formation of a train, responsible for the instability, are of the same type with a group velocity independent of the direction of the wave vector. The imaginary part of ω is assumed to be small and to depend on the direction and modulus of the real wave vector $\mathbf{k}$. In particular, such properties are possessed by perturbations of the surface of an isotropic elastic plate in a supersonic flow of a low-density gas.[9] We will choose, for example, a certain closed trajectory of the propagation of the perturbations, consisting of four sections (see the Fig. 1). In view of the specular nature of the reflection of isotropic perturbations, the trajectory will have the form of a parallelogram ABCD with sides parallel to the diagonals of the rectangle in which the perturbations are considered. We will denote the length of the sides AB, BC, CD and DA by l_1, l_2, l_3, l_4 ($l_1 = l_3$, $l_2 = l_4$). It is obvious that $l_1 + l_2 = L$, where L is the length of the diagonal of the rectangle.

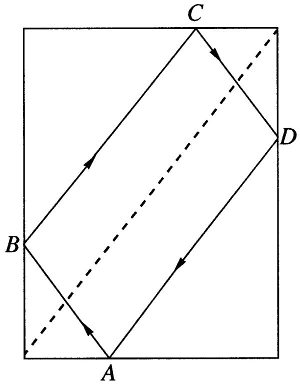

Fig. 1.

We will obtain the average increment of growth rate of the perturbations, which traverse the path represented by the parallelogram ABCD, assuming that the amplification (or attenuation) along each rectangular section occurs exponentially, according to the dispersion equation, and neglecting the difference of the reflection coefficients from unity. In the case considered, equality (1.6) can be rewritten in the form

$$\mathrm{Im}\omega = a_1(\omega_0)l_1 + a_2(\omega_0)l_2; \quad a_n(\omega_0) = -\frac{(\mathrm{Im}k_n + \mathrm{Im}k_{n+2})l_n}{2LV^{-1}}, \quad n = 1, 2 \tag{2.1}$$

Expression (2.1), taking into account the fact that $l_1 + l_2 = L$, enables us, for a specified real value of ω_0, to obtain a parallelogram with maximum growth rate of the perturbations $\mathrm{Im}\Delta\omega$. If $a_1(\omega_0) > a_2(\omega_0)$ we must put $l_1 = L$ and $l_2 = 0$ for this case and when $a_1(\omega_0) < a_2(\omega_0)$ conversely we must put $l_1 = 0$ and $l_2 = L$. Hence the optimum parallelogram when $a_1(\omega_0) \neq a_2(\omega_0)$ becomes elongated along one or other diagonal. Moreover, we can independently obtain the value of ω_0 corresponding to the maximum growth rate of the perturbations $\mathrm{Im}\omega$.

The set of parallelograms, which can represent the trajectories along which the perturbations propagate, is characterized by a single free parameter, for which we can choose the quantity l_1 or the position of one of the points of reflection on the corresponding side.

In the case considered above, the perturbation when it traverses the closed trajectory is reflected once from the boundaries of the rectangle. Other versions of closed trajectories are possible, characterised by two integers m and n, representing the number of reflections from each of the pair of parallel boundaries of the rectangle. The trajectories themselves are defined in this case, as in the case considered, to within a single parameter, which specifies, for example, the position of one of the reflection points. The trajectory always consists of two pairs of opposite directed line segments, which, taking into account the direction of the wave propagation, represent vectors, the sum of which is obviously equal to zero. If we add these vectors which are in the same direction, then, from the four sums obtained we can compile the overall parallelogram, the dimensions of which are determined by the above-mentioned numbers m and n. The growth increment of the amplitude of the perturbations ($\mathrm{Im}\omega$) along the closed trajectories can be found using equality (2.1) for the overall parallelogram. We can obviously also use equality (1.6), taking into account the fact that V_α is independent of α. As in the case considered ($m = 1$, $n = 1$), from the overall parallelograms with specified m and n we can choose the one which corresponds to the greatest value of $\mathrm{Im}\omega$. However, the possibilities for changing the lengths of the sides of the overall parallelogram for m and n greater than unity turn out to be limited. For open trajectories equality (1.4) can be used as the formula for determining the increment for sufficiently long times T.

We will present two assertions without proof, since they are intuitively obvious, while the proofs involve elementary mathematics. For specified m and n the possible versions of each of the sides of the overall parallelogram do not exceed a certain quantity, independent of m and n (for which we can choose double the length of the diagonal of the rectangle). When m and n approach infinity, the relative fraction of the path of the perturbation traversed in the specified direction approaches ¼. This assertion holds for infinite open trajectories when the section of the trajectory for which this fraction is calculated approaches infinity. If we assume that the fraction of the path traversed by the perturbation in each of the four directions is equal to ¼ (which is true for open trajectories, and also approximately true for closed trajectories with large m and n), the growth increment of the perturbations Imω will depend on a single parameter, which specifies the direction of one of the sections of the trajectory.

From the above assertions we can draw the following practical conclusion for finding the trajectory corresponding to the greatest value of Imω in the case when perturbations develop in a rectangular region. We must consider trajectories corresponding to a certain set of combinations of m and n with small values of m and n. For each combination of m and n we obtain the maximum value of Imω by varying Reω and the parameter defining the shape of the trajectory. For closed trajectories with large values of m and n, and also for open (infinite) trajectories, we can find Imω, assuming that, in each of the four directions, the perturbation covers a quarter of its path. In this case, when finding the maximum value of Imω, the parameters varied will be the ratio m/n and ω_0. Here, for closed trajectories the value of Imω obtained in this way will turn out to be only approximate, but the error will be less the greater the values of m and n.

If a trajectory with Im$\omega < 0$ is obtained by the method indicated above, we can assert that perturbations which grow with time exist if the characteristic dimension of the rectangle L is sufficiently large.

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